TWISTED K-HOMOLOGY THEORY, TWISTED Ext-THEORY

V. MATHAI AND I.M. SINGER

ABSTRACT. These are notes on twisted K-homology theory and twisted Ext-theory from the C^*-algebra viewpoint, part of a series of talks on "C^*-algebras, noncommutative geometry and K-theory", primarily for physicists.

INDEX OF NOTATION

• $\mathcal{K}$ is the algebra of compact operators on a (separable, infinite dimensional) Hilbert space $\mathcal{H}$.

• $Aut(\mathcal{K})$ is the group of automorphisms of $\mathcal{K}$.

• $PU = PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ is the group of projective unitary automorphisms of the Hilbert space $\mathcal{H}$. We will often identify PU with $Aut(\mathcal{K})$ using the canonical isomorphism between these groups.

• $\mathcal{B}$ will often denote the algebra $C_0(X, \mathcal{E}_H)$ of sections, vanishing at infinity, of the unique locally trivial bundle $\mathcal{E}_H$ over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ whose Dixmier-Douady invariant (see introduction), $\delta(\mathcal{E}_H) = [H] \in H^3(X, \mathbb{Z})$.

• $\mathcal{A}$ will often denote an algebra obtained as an extension of $\mathcal{B}$ by $\mathcal{K}$.

• P_H is the unique principal $Aut(\mathcal{K})$-bundle over X whose Dixmier-Douady invariant, $\delta(P_H) = [H] \in H^3(X, \mathbb{Z})$.

• $\mathcal{F}$ denotes the space of all Fredholm operators on a Hilbert space $\mathcal{H}$

• $\mathcal{Q} = \mathcal{Q}(\mathcal{H})$ denotes the Calkin algebra, that is $\mathcal{Q} = B(\mathcal{H})/\mathcal{K}$ where $B(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$.

V.M. acknowledges support from the Clay Mathematical Institute. I.M.S. acknowledges support from DOE contract # DE-FG02-88ER25066.

1. Introduction

Long ago, Dixmier and Douady [DD] observed that the algebra bundles with fibre $\mathcal{K}$ over a locally compact space X were classified by $[H] \in H^3(X, \mathbb{Z})$ (because $Aut(\mathcal{K}) \cong PU$ and $\pi_j(PU) = 0$ for $j \neq 2$ but $\pi_2(PU) \cong \mathbb{Z}$).

Let P_H be a principal bundle over X with fibre PU and $\delta(P_H) = [H] \in H^3(X, \mathbb{Z})$. (Transgression etc). Let $\mathcal{E}_H$ be the bundle $P_H \times_G \mathcal{K}$ and $\mathcal{F}_H$ the bundle $P_H \times_G \mathcal{F}$, where $G = PU = Aut(\mathcal{K})$, also acts on $\mathcal{K}$ by conjugation. Let $C_0(X, \mathcal{E}_H)$ be the continuous sections of $\mathcal{E}_H$ vanishing at infinity. Rosenberg [Ros] defined twisted K-theory (section 2) and showed that $\mathcal{F}_H$ is the classifying space for twisted K^0, an extension of the well known theorem for $H = 0$, i.e., $\mathcal{F}$ is the classifying space for ordinary K^0.

In recent times, twisted K-theory has entered string/M-theory when the H-field of the Neveu-Schwarz sector is turned on. See Bouwknegt-Mathai [BM], Witten [Wi], for the linking of the physics with the C^*-algebras, and the references there to twisted K-theory. There has been considerable speculation about the role for K-homology (even for C^*-algebras) in D-brane theory. See Harvey-Moore [HaMo] and the references there.

Once one is in the twisted category, one asks whether K-homology, Ext-theory, and Fredholm modules can similarly be twisted and with the usual relations between them. This is indeed the case, and is really part of Kasparov's KK-theory, a prediction made to the second author by R.G. Douglas.

These notes are a short exposition of K-homology, Ext-theory, and Fredholm modules in the twisted category from the C^*-algebra viewpoint. They are meant primarily for physicists. One can also develop a twisted theory for any generalized cohomology theory. How to do so was explained to us by M.J. Hopkins.

2. Twisted K-theory and noncommutative geometry

Let X be a locally compact, Hausdorff space with a countable basis of open sets, for example a smooth manifold. Let $[H] \in H^3(X, \mathbb{Z})$. Then the *twisted K-theory* was defined by Rosenberg as

$$K^j(X, [H]) = K_j(C_0(X, \mathcal{E}_H)) \qquad j = 0, 1, \tag{2.1}$$

where $\mathcal{E}_H$ is the unique locally trivial bundle over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ whose Dixmier-Douady invariant, $\delta(\mathcal{E}_H) = [H]$, and $K_\bullet(C_0(X, \mathcal{E}_H))$ denotes the topological K-theory of the C^*-algebra of continuous sections of $\mathcal{E}_H$ that vanish at infinity. See [Black] or [Singer] for the definition of the topological K-theory of C^*-algebras. Notice that when $H = 0$, then $\mathcal{E}_H = X \times \mathcal{K}$; therefore $C_0(X, \mathcal{E}_H) = C_0(X) \otimes \mathcal{K}$ and by Morita invariance of K-theory (cf. [Black] or [Singer]), the twisted K-theory of X coincides with the standard K-theory of X in this case. Elements of $K^0(X, [H])$ are called (virtual) gauge-bundles in the physics literature, but we will call these twisted bundles in these notes.

In [Ros], it is shown that when X is compact one has

$$\begin{aligned} K^0(X, [H]) &= [P_H, \mathcal{F}]^{PU} \\ K^1(X, [H]) &= [P_H, U(\mathcal{K}^+)]^{Aut(\mathcal{K})} \end{aligned} \tag{2.2}$$

where $U(\mathcal{K}^+)$ is the group of unitaries in the unitalization of $\mathcal{K}$,

$$U(\mathcal{K}^+) = \{u \in U(\mathcal{H}) : u - 1 \in \mathcal{K}\}\,.$$

3. The twisted Ext group and twisted K-homology

In this section we will give a brief review of the twisted *Ext* group, which can be considered as a specialization of the general *Ext*-theory [PPV], [Kas].

Consider noncommutative C^* algebras $\mathcal{A}$ which fit into the short exact sequence:

$$0 \to \mathcal{K} \to \mathcal{A} \xrightarrow{\beta} \mathcal{B} \to 0 \tag{3.1}$$

where $\mathcal{B} = C_0(X, \mathcal{E}_{[H]})$, for some fixed space X and NS field H. In [PPV], [Kas] extensions of the form (3.1) for general nuclear C^*-algebras $\mathcal{B}$ were investigated. We shall restrict ourselves to the special case when $\mathcal{B} = C_0(X, \mathcal{E}_{[H]})$. To any such extension one can associate the Busby invariant, which is a homomorphism

$$\tau : C_0(X, \mathcal{E}_{[H]}) \to Q(\mathcal{H}) \tag{3.2}$$

defined as follows. For any $s \in C_0(X, \mathcal{E}_{[H]})$ we choose an operator $T_s \in \mathcal{A}$ such that $\beta(T_s) = s$, and define τ by: $\tau(s) = \pi(T_s)$ where $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ is the projection. τ is a homomorphism because $T_{s_1}T_{s_2} - T_{s_1 s_2}$ is a compact operator. Conversely, given a homomorphism $\tau : C_0(X, \mathcal{E}_{[H]}) \to Q(\mathcal{H})$ one can form an extension 3.1 as follows,

$$0 \to \mathcal{K} \to \mathcal{A}' \ \to \ C_0(X, \mathcal{E}_{[H]}) \to 0 \tag{3.3}$$

where the algebra $\mathcal{A}'$ is defined as

$$\mathcal{A}' = \{(A, f) : \pi(A) = \tau(f)\} \subset B(\mathcal{H}) \oplus C_0(X, \mathcal{E}_{[H]}). \tag{3.4}$$

Moreover, (3.1) is unitarily equivalent to (3.3) in the sense that we now describe.

Two extensions (3.1) are unitarily equivalent if there is a unitary operator U on $\mathcal{H}$ such that the Busby invariants are related by $\tau_2(s) = \pi(U)\tau_1(s)\pi(U)^*$. Let $\mathbf{Ext}(X, H)$ denote the set of unitary equivalence classes of extensions of $C_0(X, \mathcal{E}_{[H]})$ by $\mathcal{K}$. A direct sum operation on $\mathbf{Ext}(X, [H])$ can then be defined by taking the extension corresponding to the Busby invariant

$$\tau_1 \oplus \tau_2 : C_0(X, \mathcal{E}_{[H]}) \to Q(\mathcal{H}) \oplus Q(\mathcal{H}) \to Q(\mathcal{H} \oplus \mathcal{H}) \cong Q(\mathcal{H}). \tag{3.5}$$

Then (3.5) defines a semigroup operation on $\mathbf{Ext}(X, H)$. Trivial extensions are those for which the Busby invariant lifts to $B(\mathcal{H})$. Equivalently, they are extensions such that the sequence (3.1) splits. Define the *twisted Ext group* $Ext(X,[H])$ as being the quotient of $\mathbf{Ext}(X,H)$ by the trivial extensions. It is shown in [PPV], [Kas] that every extension has an inverse up to the addition of a trivial extension, so that $Ext(X,[H])$ is an abelian group. It is clear that $Ext(X,[H])$ depends only on the cohomology class of $[H]$, since $C_0(X,\mathcal{E}_{[H]})$ and $C_0(X,\mathcal{E}_{[H']})$ are isomorphic whenever $[H']=[H]$.

There is a pairing of $Ext(X,[H])$ and $K^1(X,[H])$ defined as follows. (See [BM], [DK], [Ka], [Ros], [Wi] for a discussion of twisted K-theory, $K^\bullet(X,[H])$). An extension of the form (3.1) is determined by its Busby invariant $\tau : C_0(X,\mathcal{E}_{[H]}) \to Q(\mathcal{H})$, which induces homomorphisms $\widetilde{\tau} : M_n(C_0(X,\mathcal{E}_{[H]})^+) \to M_n(Q(\mathcal{H})) \cong Q(\mathcal{H})$ for each $n \in \mathbb{N}$. If u is a unitary in $M_n(C_0(X,\mathcal{E}_{[H]})^+)$, define the pairing

$$
\begin{aligned}
Ext(X,[H]) \times K^1(X,[H]) &\to \mathbb{Z} \\
(\tau,u) &\to \mathrm{Index}(\widetilde{\tau}(u)).
\end{aligned}
\tag{3.6}
$$

In particular, each element $\tau \in Ext(X,[H])$ defines a homomorphism $\tau_* : K^1(X,[H]) \to \mathbb{Z}$. If $\tau_*=0$, then the six term exact sequence in K-theory (cf. section 4) corresponding to the extension (3.1) reduces to the short exact sequence

$$
0 \to \mathbb{Z} = K_0(\mathcal{K}) \to K_0(\mathcal{A}) \to K^0(X,[H]) \to 0 \tag{3.7}
$$

and therefore defines an element of $Ext^1_{\mathbb{Z}}(K^0(X,[H]),\mathbb{Z})$ in homological agebra. It can be shown [RS] that the converse is also true, that is, one has the universal coefficient theorem:

$$
0 \to Ext^1_{\mathbb{Z}}(K^0(X,[H]),\mathbb{Z}) \to Ext(X,[H]) \to Hom(K^1(X,[H]),\mathbb{Z}) \to 0. \tag{3.8}
$$

This justifies the definition of the twisted K-homology as being

$$
\begin{aligned}
K_1(X,[H]) &= Ext(X,[H]) \\
K_0(X,[H]) &= Ext(X\times\mathbb{R}, p_1^*[H])
\end{aligned}
\tag{3.9}
$$

where $p_1 : X\times\mathbb{R} \to X$ denotes projection onto the first factor.

One deduces from the definition (3.9) that the universal coefficient exact sequence (3.8) can be rewritten as

$$0 \to Ext^1_{\mathbb{Z}}(K^{\bullet+1}(X,[H]),\mathbb{Z}) \to K_\bullet(X,[H]) \to Hom(K^\bullet(X,[H]),\mathbb{Z}) \to 0. \tag{3.10}$$

4. Properties of the twisted Ext groups and twisted K-homology

1) **Bott Periodicity:** Let $[H] \in H^3(X,\mathbb{Z})$. Then one has the Bott periodicity theorem for the C^*-algebra $C_0(X,\mathcal{E}_H)$,

$$Ext(X\times\mathbb{R}^2, p_1^*[H]) \cong Ext(X,[H]) \tag{4.1}$$

For details on Bott periodicity for C^*-algebras, cf. [Black] or [Singer].

This shows that if we define $K_j(X,[H]) = Ext(X\times\mathbb{R}^j, p_1^*[H])$ for $j\in\mathbb{N}$, then there are at most only two distinct groups in this list, $K_1(X,[H])$ and $K_0(X,[H])$.

2) **Six term exact sequence:** Given a short exact sequence

$$0 \to C_0(X,\mathcal{E}_{[H]}) \to \mathcal{B} \to \mathcal{J} \to 0$$

there is a six term exact sequence of twisted Ext groups, which is obtained using Bott periodicity,

$$\begin{array}{ccccc}
K^0(\mathcal{J}) & \longrightarrow & K^0(\mathcal{B}) & \longrightarrow & K_0(X,[H]) \\
\delta\big\uparrow & & & & \delta\big\downarrow \\
K_1(X,[H]) & \xleftarrow{F_*} & K^1(\mathcal{B}) & \xleftarrow{\Delta} & K^1(\mathcal{J})
\end{array}$$

This enables us to compute the twisted Ext groups at least in some examples. Consider the case when $X = S^3$ and $[H]$ is the class of the volume form on S^3 and N is a positive integer. Then we will compute $K_\bullet(S^3, N[H])$. Note that this can also be done using the universal coefficient theorem.

We consider the open cover of S^3 given by the upper and lower hemispheres, $\{\mathcal{U}_1,\mathcal{U}_2\}$, where $\mathcal{U}_1\cap\mathcal{U}_2 = S^2$. Then representatives of $K^1(S^3,N[H])$ are pairs of maps (f_1, f_2), where

$$f_i : \mathcal{U}_i \to U(\mathcal{K}^+)$$

such that on the overlap $\mathcal{U}_1 \cap \mathcal{U}_2 = S^2$, one has

$$f_1 = p_{N[H]} f_2, \tag{4.2}$$

where $p_{N[H]}$ denotes the transition functions of the bundle $\mathcal{K}$-algebra bundle $\mathcal{E}_{N[H]}$ with Dixmier-Douady invariant $N[H] \in H^3(S^3, \mathbb{Z})$. Now the C^*-algebra of continuous sections of the bundle $\mathcal{E}_{N[H]}$, $C(X, \mathcal{E}_{N[H]})$ can be represented by pairs of continuous functions (h_1, h_2), where

$$h_i : \mathcal{U}_i \to \mathcal{K} \qquad i = 1, 2$$

and satisfying on the overlap $\mathcal{U}_1 \cap \mathcal{U}_2 = S^2$

$$h_1 = p_{N[H]} h_2.$$

Therefore there is a short exact sequence

$$0 \to C(S^3, \mathcal{E}_{N[H]}) \xrightarrow{F} C(\mathcal{U}_1) \otimes \mathcal{K} \oplus C(\mathcal{U}_2) \otimes \mathcal{K} \xrightarrow{G} C(S^2) \otimes \mathcal{K} \to 0 \tag{4.3}$$

where

$$F(h_1, h_2) = h_1 \oplus h_2, \qquad G(q_1 \oplus q_2) = q_1|_{S^2} - p_{N[H]}\left(q_2|_{S^2}\right).$$

The six term exact sequence in *Ext*-theory associated to the short exact sequence (4.3) is,

$$\begin{array}{ccccc} K_0(S^2) & \xrightarrow{G^*} & K_0(\mathcal{U}_1) \oplus K_0(\mathcal{U}_2) & \xrightarrow{F^*} & K_0(S^3, N[H]) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1(S^3, N[H]) & \xleftarrow{F^*} & K_1(\mathcal{U}_1) \oplus K_1(\mathcal{U}_2) & \xleftarrow{G^*} & K_1(S^2) \end{array}$$

Since $0 = K_1(\mathcal{U}_1) \oplus K_1(\mathcal{U}_2) = K_1(S^2)$, this six term exact sequence collapses into the exact sequence

$$0 \to K_1(S^3, N[H]) \xrightarrow{\delta} K_0(S^2) \xrightarrow{G^*} K_0(\mathcal{U}_1) \oplus K_0(\mathcal{U}_2) \xrightarrow{F^*} K_0(S^3, N[H]) \to 0 \tag{4.4}$$

On analyzing the map G^* explicitly, we see that if $N \neq 0$, then $Ker(G^*) = 0 = K_1(S^3, N[H])$ and that $Coker(G^*) = \mathbb{Z}_N \cong K_0(S^3, N[H])$. See [Ros] and [BM] for related computations.

3) **When H is torsion:** In this case, there is an argument in [Gr],[Wi] which shows that the free part of $K^0(X, [H])$ is isomorphic to the free part of $K^0(X)$. Therefore $Hom(K^0(X, [H]), \mathbb{Z}) \cong Hom(K^0(X), \mathbb{Z})$, and by the universal coefficient theorem, we see that the free part of the twisted K-homology $K_0(X, [H])$ is isomorphic to the free part of $K_0(X)$ in this case, and in particular, $K_0(X, [H]) \otimes \mathbb{Q} \cong K_0(X) \otimes \mathbb{Q}$.

4) **The fundamental class when $H = 0$:** A noteworthy case, to be reviewed later in these talks, is the extension

$$0 \to \mathcal{K} \to \mathcal{A} \to C(S^*X) \to 0 \tag{4.5}$$

where $\mathcal{A}$ is the closure in the norm topology of the algebra of singular integral operators (pseudodifferential operators of order zero) and S^*X is the sphere bundle of the cotangent bundle of a smooth manifold X. The extension (4.5) does not split, and is called the fundamental class in $Ext(S^*X)$, cf. [Kas], [BD].

5. Description of twisted K-homology in terms of Fredholm operators

We recall here the definition of $KK^1(\mathcal{B}, \mathbb{C})$, where $\mathcal{B} = C_0(X, \mathcal{E}_{[H]})$ and $\mathbb{C}$ is the algebra of complex numbers, a very special case of Kasparov's $KK(\mathcal{B}, \mathcal{D})$ theory for general C^*-algebras $\mathcal{D}$. This will provide us with a Fredholm module picture for twisted Ext-theory. A Fredholm module is a triple $(\mathcal{H}, \phi, F)$, where,

- $\mathcal{H}$ is a separable Hilbert space;
- $\phi : C_0(X, \mathcal{E}_{[H]}) \to B(\mathcal{H})$ is a $*$-homomorphism;

- F is self-adjoint and satisfies: $(F^2 - 1)\phi(a) \in \mathcal{K}$, and $[F, \phi(a)] \in \mathcal{K}$ for all $a \in \mathcal{B}$.

Let $\mathrm{E}_1(\mathcal{B})$ denote the set of all Fredholm modules over $\mathcal{B}$. Let $\mathrm{D}_1(\mathcal{B})$ denote the subset of Fredholm modules satisfying $(F^2-1)\phi(a) = 0 = [F, \phi(a)]$. They are called degenerate Fredholm modules.

The direct sum of two Fredholm modules is again a Fredholm module. Moreover, the direct sum of degenerate Fredholm modules is again a degenerate Fredholm module. Two Fredholm modules $(\mathcal{H}_i, \phi_i, F_i),\ i = 0, 1$ are said to be unitarily equivalent if there is a unitary in $B(\mathcal{H}_0, \mathcal{H}_1)$ intertwining the ϕ_i and the F_i.

Define an equivalence relation $\sim$ on $\mathrm{E}(\mathcal{B})$ generated by unitary equivalence, addition of degenerate elements and 'compact perturbations' of $(\mathcal{H}, \phi, F)$. Here a Fredholm module $(\mathcal{H}, \phi, F')$ is said to be a compact perturbation of $(\mathcal{H}, \phi, F)$ if $(F - F')\phi(a) \in \mathcal{K}$ for all $a \in \mathcal{B}$.

Then $KK^1(\mathcal{B}, \mathbb{C})$ is the set of equivalence classes of $\mathrm{E}_1(\mathcal{B})$ under the equivalence relation $\sim$.

Given a Fredholm module $(\mathcal{H}, \phi, F)$, we will define a $\mathcal{K}$ extension of $\mathcal{B}$ of the form (3.1) as follows. Observe that $P = 1/2F + 1/2$ is a projection modulo $\mathcal{K}$. Define the Busby map τ by $\tau(a) = \pi(P\phi(a)P)$ for all $a \in \mathcal{B}$, where $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ is the projection. Then τ gives the desired $\mathcal{K}$ extension of $\mathcal{B}$ of the form (3.1). The Busby map corresponding to $1 - P$ is an inverse for τ, and we have a well defined map

$$KK^1(\mathcal{B}, \mathbb{C}) \to Ext(X, [H]) = K_1(X, [H]).$$

It is not too hard to show that this map is an isomorphism

$$KK^1(\mathcal{B}, \mathbb{C}) \cong Ext(X, [H]) = K_1(X, [H]).$$

This gives a Fredholm module description of twisted Ext-theory, or equivalently of twisted K-homology theory.

There is also a $\mathbb{Z}_2$-graded Fredholm module description of the twisted K-homology group $K_0(X,[H])$, which we will now discuss. A $\mathbb{Z}_2$-graded Fredholm module is a triple $(\mathcal{H},\phi,F)$, where $\mathcal{H}$ is a separable $\mathbb{Z}_2$-graded Hilbert space, $\phi : C_0(X,\mathcal{E}_{[H]}) \to B(\mathcal{H})$ is a $*$-homomorphism which is of even degree, F is an odd degree self-adjoint operator on $\mathcal{H}$ and satisfies $(F^2-1)\phi(a) \in \mathcal{K}, \quad [F,\phi(a)] \in \mathcal{K}$ for all $a \in \mathcal{B}$. Let $\mathrm{E}_0(\mathcal{B})$ denote the set of all $\mathbb{Z}_2$-graded Fredholm modules over $\mathcal{B}$. Let $\mathrm{D}_0(\mathcal{B})$ denote the subset of $\mathbb{Z}_2$-graded Fredholm modules satisfying $(F^2-1)\phi(a) = 0 = [F,\phi(a)]$. They are called degenerate $\mathbb{Z}_2$-graded Fredholm modules.

The direct sum of two $\mathbb{Z}_2$-graded Fredholm modules is again a $\mathbb{Z}_2$-graded Fredholm module, with respect to the total $\mathbb{Z}_2$-grading. Moreover, the direct sum of degenerate $\mathbb{Z}_2$-graded Fredholm modules is again a degenerate $\mathbb{Z}_2$-graded Fredholm module. Two $\mathbb{Z}_2$-graded Fredholm modules $(\mathcal{H}_i,\phi_i,F_i), \quad i=0,1$ are said to be unitarily equivalent if there is a unitary in $B(\mathcal{H}_0,\mathcal{H}_1)$ intertwining the ϕ_i and the F_i.

Define an equivalence relation $\sim$ on $\mathrm{E}_0(\mathcal{B})$ generated by unitary equivalence, addition of degenerate elements and 'compact perturbations' of $(\mathcal{H},\phi,F)$. Here a $\mathbb{Z}_2$-graded Fredholm module $(\mathcal{H},\phi,F')$ is said to be a compact perturbation of $(\mathcal{H},\phi,F)$ if $(F-F')\phi(a) \in \mathcal{K}$ for all $a \in \mathcal{B}$.

Then $KK^0(\mathcal{B},\mathbb{C})$ is the set of equivalence classes of $\mathrm{E}_0(\mathcal{B})$ under the equivalence relation $\sim$. It follows from the discussion above and Bott periodicity that
$KK^0(\mathcal{B},\mathbb{C}) = KK^1(\mathcal{B}\otimes C_0(\mathbb{R}),\mathbb{C}) \cong Ext(X\times\mathbb{R},p_1^*[H]) = K_0(X,[H])$.

6. Topological K-homology

We now give a Baum-Douglas type description of twisted K-homology, called topological twisted K-homology. The basic objects are twisted K-cycles. A *twisted*

K-cycle on a topological space is a triple (M, E, ϕ), where M is a compact $\mathrm{Spin}^{\mathbb{C}}$ manifold, $E \to M$ is a twisted bundle on M, and $\phi : M \to X$ is a continuous map. Two twisted K-cycles (M, E, ϕ) and (M', E', ϕ') are said to be *isomorphic* if there is a diffeomorphism $h : M \to M'$ such that $h^*(E') \cong E$ and $h^*\phi' = \phi$. Let $\Pi(X, H)$ denote the collection of all twisted K-cycles on X.

• *Bordism*: $(M_i, E_i, \phi_i) \in \Pi(X, H)$, $i = 0, 1$ are said to be *bordant* if there is a triple (W, E, ϕ) where W is a compact $\mathrm{Spin}^{\mathbb{C}}$ manifold with boundary ∂W, E is a twisted bundle over W and $\phi : W \to X$ is a continuous map, such that $(\partial W, E\big|_{\partial W}, \phi\big|_{\partial W})$ is isomorphic to the disjoint union $(M_0, E_0, \phi_0) \cup (-M_1, E_1, \phi_1)$. Here $-M_1$ denotes M_1 with the reversed $\mathrm{Spin}^{\mathbb{C}}$ structure.

• *Direct sum*: Suppose that $(M, E, \phi) \in \Pi(X, H)$ and that $E = E_0 \oplus E_1$. Then (M, E, ϕ) is isomorphic to $(M, E_0, \phi) \cup (M, E_1, \phi)$.

• *Twisted bundle modification*: Let $(M, E, \phi) \in \Pi(X, H)$ and $\mathbf{H}$ be an even dimensional $\mathrm{Spin}^{\mathbb{C}}$ vector bundle over M. Let $\widehat{M} = S(\mathbf{H} \oplus 1)$ denote the sphere bundle of $\mathbf{H} \oplus 1$. Then $\widehat{M}$ is canonically a $\mathrm{Spin}^{\mathbb{C}}$ manifold. Let $\mathcal{S}$ denote the bundle of spinors on $\mathbf{H}$. Since $\mathbf{H}$ is even dimensional, $\mathcal{S}$ is $\mathbb{Z}_2$-graded,

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$$

into bundles of 1/2-spinors on M. Define $\widehat{E} = \pi^*(\mathcal{S}^{+*} \otimes E)$, where $\pi : \widehat{M} \to M$ is the projection. Finally, $\widehat{\phi} = \pi^*\phi$. Then $(\widehat{M}, \widehat{E}, \widehat{\phi}) \in \Pi(X, H)$ is said to be obtained from (M, E, ϕ) and $\mathbf{H}$ by *twisted bundle modification*.

Let $\sim$ denote the equivalence relation on $\Pi(X, H)$ generated by the operations of bordism, direct sum and twisted bundle modification. Notice that $\sim$ preserves the parity of the dimension of the twisted K-cycle. Let $K_0^t(X, [H])$ denote the quotient $\Pi^{even}(X, H)/\sim$, where $\Pi^{even}(X, H)$ denotes the set of all even dimensional twisted K-cycles in $\Pi(X, H)$, and let $K_1^t(X, [H])$ denote the quotient $\Pi^{odd}(X, H)/\sim$, where $\Pi^{odd}(X, H)$ denotes the set of all odd dimensional twisted K-cycles in $\Pi(X, H)$.

Then it is possible to show as in [BD] that $K_j^t(X,[H]) \cong K_j(X,[H]),\ j=0,1$, providing a topological description of twisted K-homology.

References

[BD] P. Baum and R. Douglas, K homology and index theory, Operator algebras and applications, Part I (Kingston, Ont., 1980), pp. 117-173, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.

[Black] B. Blackadar, K-theory for operator algebras, MSRI publications, vol 5, Cambridge University Press, 1986.

[BM] P. Bouwknegt and V. Mathai, *D-Branes, B-Fields and twisted K-theory*, Journal of High Energy Physics, **03** (2000) 007 (11 pages); *ibid.*, in preparation.

[DD] J. Dixmier and A. Douady, *Champs continues d'espaces hilbertiens at de C^*-algèbres*, Bull. Soc. Math. France **91** (1963) 227–284.

[DK] P. Donovan and M. Karoubi, *Graded Brauer groups and K-theory with local coefficients*, Inst. Hautes Études Sci. Publ. Math., **38** (1970) 5–25.

[Gr] A. Grothendieck, *Le groupe de Brauer, I, II, III. 1968 Dix Exposés sur la Cohomologie des Schémas*, pp. 46–188, (North-Holland, Amsterdam; Masson, Paris).

[HaMo] J. Harvey, G. Moore, Noncommutative Tachyons and K-Theory, [`hep-th/0009030`]

[Ka] A. Kapustin, *D-branes in a topologically nontrivial B-field*, [`hep-th/9909089`].

[Kas] G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Inv. Math.* **91** (1988), 147-201. *ibid.*, Topological invariants of elliptic operators, I. K-homology. (Russian) Math. USSR-Izv. **9** (1975), no. 4, 751-792.

[PPV] M. Pimsner, S. Popa, D. Voiculescu, Homogeneous C^*-extensions of $C(X) \otimes K(H)$. II. J. Operator Theory **4** (1980), no. 2, 211-249; *ibid*, Homogeneous C^*-extensions of $C(X) \otimes K(H)$. I. J. Operator Theory **1** (1979), no. 1, 55-108.

[Ros] J. Rosenberg, *Continuous trace algebras from the bundle theoretic point of view*, Jour. Austr. Math. Soc., **47** (1989), 368–381; *ibid.*, *Homological invariants of extensions of C^*-algebras*, Proceedings of Symposia in Pure Mathematics, **38** (1982) 35-75.

[RS] J. Rosenberg and C. Schochet, The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, *Duke Math. J.* **55** (1987), no. 2, 431-474.

[Singer] V. Mathai and I.M. Singer, Lectures on operator algebras, noncommutative geometry and K-theory (primarily for physicists), fall 2000, MIT (in preparation).

[Wi] E. Witten, *D-branes and K-theory*, JHEP **12** (1998) 019, [hep-th/9810188]; *ibid.*, Overview Of K-Theory Applied To Strings, [hep-th/0007175].

Department of Mathematics, MIT, Cambridge, Mass 02139, USA and University of Adelaide, Adelaide 5005, Australia

Department of Mathematics, MIT, Cambridge MA 02139